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# Identification of the Radiative and Nonradiative Parts of a Wave Field

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We present a method for decomposing a wave field, described by a second-order ordinary differential equation, into a radiative component and a nonradiative one, using a biorthonormal system related to the problem under consideration. We show that it is possible to select a special system such that the wave field is purely radiating. We discuss the differences and analogies with approaches which, unlike our approach, start from the corresponding sources of the field.

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**1. Introduction.**—The identification of the radiative and nonradiative parts of a field or source distribution has received much attention recently. In the present paper we have in mind the radiation produced by a charge-current distribution. However, the considerations in this Letter apply to any linear inhomogeneous problem such as, e.g., the transport and diffusion equation [1].

In [2] we considered the problems for the fields produced by a source distribution while Marengo and Ziolkowski [3] studied the corresponding problem for sources. The problem arises in a natural way when one wants to know the solution of the inverse problem of determining the source distribution from the measured field: could it be that different source distributions give rise to the same observed field? Or formulated differently: do source distributions exist which do not give rise to observable fields, so that source distributions cannot be reconstructed uniquely from the measured field data? We will discuss this problem in the context of a scalar spherically symmetric source distribution which is confined to a spherical region  $D$ . The generalization to more general situations should be evident, albeit at the expense of more complicated mathematics.

We focus our discussion on the fields generated by the source distribution. We prefer this choice because nonradiating fields have been characterized by Kim and Wolf [4] and Gamliel *et al.* [5] as those fields  $u(r)$  which vanish outside  $D$  and which satisfy

$$u(R) = 0 \quad \text{and} \quad [\partial u / \partial r]_{r=R} = 0$$

on the boundary  $\partial D$  of  $D$ . (1)

The basic tool in our approach is a biorthonormal system which is closely related to the problem: one component is a set of inhomogeneous nonradiating modes satisfying the boundary conditions (1) into which the nonradiating component of the field can be expanded, while the other component is a set of homogeneous modes which might be useful for the identification of the radiative component.

In Section 2 we state the problem and introduce the inhomogeneous modes along the lines of [2]. We identify the nonradiative component as the projection of the field on the space spanned by the inhomogeneous modes. The corre-

sponding result is derived in Section 3. The basic result of this section is that the apparently complete freedom of the choice for the “arbitrary” charge distribution generating the inhomogeneous modes is not existent, and that this charge distribution is, in fact, uniquely determined by the physics of the problem. In an analogous way we identify in Section 4 the radiative component as the projection of the field on the space spanned by the homogeneous modes. We show that this definition in spite of its attractiveness leads to problems.

A related but basically different approach has been given by Gamliel *et al.* [5] who construct nonradiative functions from homogeneous modes. See also the papers by Berry *et al.* [6] and Gbur *et al.* [7].

We consider a very simple situation: the equation describing the generation of waves is a second-order ordinary differential equation. This seems an oversimplification, but more general cases can be discussed along similar lines, and only the mathematics becomes more involved. See the recent paper by Marengo and Ziolkowski [8].

**2. Formulation of the problem.**—Our basic equation is

$$\frac{d^2 u(r, k)}{dr^2} + k^2 u(r, k) = \rho_t(r) \quad (2)$$

with the boundary conditions  $u(0, k) = 0$  and the Sommerfeld radiation condition  $\frac{du(r, k)}{dr} - iku(r, k) \rightarrow 0$  if  $r \rightarrow \infty$ .

This equation can be obtained (see, e.g., [2]) from the Helmholtz equation with a spherically symmetric source  $s(r)$ . We restrict ourselves to spherically symmetrical fields  $\psi(r, k)$ : (prior information) satisfying  $(\nabla^2 + k^2)\psi(r, k) = s(r)$ . Using spherical coordinates this equation can be written as  $\frac{d^2(r\psi)}{dr^2} + k^2(r\psi) = rs(r)$ . Identifying  $r\psi(r, k) = u(r, k)$  and  $rs(r) = \rho_t(r)$  we obtain (2). The boundary condition  $u(0, k) = 0$  is satisfied because  $\psi(0, k)$  is finite while the Sommerfeld radiation condition for  $\psi(r, k)$  entails that for  $u(r, k)$ . Let us first make some remarks about the notation: for reasons which become clear very soon it is expedient to denote explicitly the dependence of  $u(r, k)$  on  $k$ . The subscript “ $t$ ” at  $\rho$  denotes that we deal with the “true” (physically present) source distribution. This seems a superfluous remark, but we will also have to deal with arbitrary source distributions

which we will give a subscript “ $a$ .” All the source distributions are confined to the same domain  $D$ :  $r \leq R$ . We assume that there are no singularities like surface source distributions.

It should be noted that  $s(r)$ , being the temporal Fourier transform of a time dependent source distribution, depends on  $k$  as well. For ease of notation we refrain from explicitly giving the  $k$  dependence of  $s(r)$  and related quantities like  $\rho_t(r)$  and  $\rho_a(r)$ .

The solution of (2) satisfying the boundary conditions is given by

$$u(r, k) = -\frac{\sin(kr)}{k} \int_0^R \rho_t(r') \exp(ikr') dr' + \frac{1}{k} \int_0^r \rho_t(r') \sin[k(r - r')] dr'. \quad (3)$$

Nonradiating fields are characterized by  $u(R, k) = [du(r, k)/dr]_{r=R} = 0$ . Applying these conditions to (3) we find for the nonradiating field  $u_{nr}(r, k)$ :

$$u_{nr}(r, k) = -\frac{C(k)}{k} \sin(kr) + \frac{1}{k} \int_0^r \rho_t(r') \sin[k(r - r')] dr', \quad (4)$$

provided  $S(k) = 0$ .

We abbreviated

$$C(k) = \int_0^R \rho_t(r') \cos(kr') dr', \quad (5)$$

$$S(k) = \int_0^R \rho_t(r') \sin(kr') dr'. \quad (6)$$

It is clear that the condition  $S(k) = 0$  can be satisfied only for discrete values  $k_n$  of  $k$ :

$$S(k_n) = 0. \quad (7)$$

We are now in a position to define a set of discrete inhomogeneous modes denoted by  $u_n(r)$  which satisfy

$$\frac{d^2 u_n}{dr^2} + k_n^2 u_n(r) = \rho_a(r), \quad (8)$$

where  $k_n$  is defined by

$$S_a(k_n) = \int_0^R \rho_a(r') \sin(k_n r') dr' = 0. \quad (9)$$

Thus the functions  $u_n(r)$  satisfy the conditions (1) for a nonradiating field. Here we draw attention to an essential point in our study: for the characterization of the nonradiating functions we need only the boundary conditions for  $u$  and  $du/dr$  on the boundary  $\partial D$  of  $D$ . We do not have to use the physically present source distribution  $\rho_t(r)$ ; we may use any source distribution  $\rho_a(r)$  provided this source is confined to the same domain  $D$  as  $\rho_t(r)$ .

The set of inhomogeneous modes has been shown by Hilb [9] to be complete: all nonradiating fields can be expanded in this set of  $u_n(r)$ . When  $\rho_a(r)$  is changed, the

space spanned by the set  $u_n(r)$  also changes correspondingly. So now we have many different complete sets  $u_n(r)$  in which the nonradiating fields can be expanded.

We now define the nonradiative part of the field  $u(r, k)$  as the projection of this field on the space spanned by the inhomogeneous modes corresponding to an arbitrary source distribution  $\rho_a(r)$ :

$$u_{nr}(r, k) = \sum_n u_n(r) a_n(k), \quad (10)$$

where  $a_n(k)$  can be found using the biorthonormal system  $u_n, v_n$  discussed in [2].  $v_n(r)$  satisfies the homogeneous equation

$$\frac{d^2 v_n}{dr^2} + k_n^2 v_n = 0 \quad (11)$$

with the boundary condition  $v_n(0) = 0$ , leading to

$$v_n(r) = -\frac{2k_n}{S'(k_n)} \sin(k_n r). \quad (12)$$

The numerical factor in front of  $\sin(k_n r)$  ensures the biorthonormality of  $u_n, v_n$ :  $\int_0^R u_n(r) v_m(r) dr = \delta_{nm}$ . Consequently we find for  $a_n(k)$

$$a_n(k) = \int_0^R u(r, k) v_n(r) dr, \quad (13)$$

where the first factor in the integrand,  $u(r, k)$ , expresses the projection of  $u(r, k)$ . In the next sections we will discuss various definitions for the radiative and nonradiative part of a wave field. The great variety of  $u_n, v_n$  opens the possibility to choose a special one such that  $u_{nr}(r, k)$  has a special property such as  $u_{nr}(r, k) = 0$ . This will be the case if  $v_n(r)$  is chosen such that  $a_n(k) = 0$ . This will be discussed in Section 3.

**3. Identification of the nonradiative component according to [2].**—The solution  $u(r, k)$  of (2) is given by (3) in which the true source distribution  $\rho_t(r)$  occurs. The nonradiative part  $u_{nr}(r, k)$  of this function is defined by (10) with  $a_n(k)$  given by (13). The radiative component of  $u(r, k)$  is defined by

$$u_{\text{rad}}(r, k) = u(r, k) - u_{nr}(r, k). \quad (14)$$

Using (10) and (13) and using the biorthonormality of  $u_n, v_n$  we straightforwardly find

$$\int_0^R u_{\text{rad}}(r, k) v_n(r) dr = 0 \quad (15)$$

with  $v_n$  given by (12).

We will now show that (15) leads to the following remarkable conclusion:

$$u_{\text{rad}}(r, k) = \rho_a(r) \quad \text{for } r < R. \quad (16)$$

To prove this we consider the following contour integral:

$$I(a) = \frac{1}{2\pi i} \oint_C \frac{\int_0^R u_{\text{rad}}(r, k) \sin(wr) dr}{(w - a) \int_0^R \sin(wr) \rho_a(r) dr} dw, \quad (17)$$

where  $C$  is the circle  $|w| = W$  in the complex  $w$  plane.  $a$  is an arbitrary complex number, not coinciding with zeros  $k_n$  of  $\int_0^R \sin(wr) \rho_a(r) dr$ . The integral  $I$  can be obtained using the calculus of residues:

$$I(a) = \frac{\int_0^R u_{\text{rad}}(r, k) \sin(ar) dr}{\int_0^R \rho_a(r) \sin(ar) dr} + \sum_n \frac{\int_0^R u_{\text{rad}}(r, k) \sin(k_n r) dr}{(k_n - a) k_n \int_0^R \cos(k_n r) \rho_a(r) dr}. \quad (18)$$

In the limit  $W \rightarrow \infty$ , the integral  $I$  can also be obtained from the asymptotic expressions for the factors of its integrand. All we need is the following asymptotic expression:

$$\int_0^R \sin(wr) f(r) dr \asymp -\frac{\cos(wR)}{w} f(R) \{1 + O(w^{-1})\}, \quad (19)$$

where  $f(r)$  is an arbitrary function. Using (19) in (17) we find

$$I(a) = \frac{u_{\text{rad}}(R, k)}{\rho_a(R)} = \frac{u(R, k)}{\rho_a(R)}. \quad (20)$$

Combining the results (15), (18), and (20) we find

$$\int_0^R \sin(ar) \left\{ u_{\text{rad}}(r, k) - \frac{u(R, k)}{\rho_a(R)} \rho_a(r) \right\} dr = 0, \quad (21)$$

which is valid for (almost) every value of  $a$ . Consequently,

$$u_{\text{rad}}(r, k) = \frac{u(R, k)}{\rho_a(R)} \rho_a(r) \quad \text{for } r < R, \quad (22)$$

as may be shown by multiplying (21) by  $e^{iap}$ , where  $p$  is an arbitrary positive number and integrating over  $a$  from  $-\infty$  to  $+\infty$ .

We derived the result (22) in a different way in [2], Eq. (48).

As  $\rho_a(r)$  is an arbitrary source, we see that the splitting of  $u(r, k)$  in radiative and nonradiative parts is arbitrary as well. So we have to pose additional conditions to enforce uniqueness. In [2] we tacitly imposed the condition  $\rho_a = \rho_t$  which misses a compelling physical motivation.

We now consider the question whether it is possible to choose  $\rho_a(r)$  such that the physical source distribution  $\rho_t(r)$  is purely radiating, i.e., lacks a nonradiating part:  $\rho_{nr}(r) = 0$ . As  $\rho_{nr}(r) = Lu_{nr}(r, k)$ , where  $L$  is the operator  $d^2/dr^2 + k^2$ , we get the equation  $Lu_{nr}(r, k) = 0$  which together with the boundary conditions (1) leads to  $u_{nr}(r, k) = 0$ . So the radiative part of  $u(r, k)$ ,  $u_{\text{rad}}(r, k)$ , equals

$$u_{\text{rad}}(r, k) = u(r, k) = \frac{u(R, k)}{\rho_a(R)} \rho_a(r) \quad \text{for } r < R, \quad (23)$$

where we used (22) and the fact that  $u_{nr}(r, k) = 0$ . Applying the operator  $L$  to (23) we find

$$\rho_{\text{rad}}(r) = \rho_t(r) = \frac{u(R, k)}{\rho_a(R)} L \rho_a(r) \quad \text{for } r < R, \quad (24)$$

which is a differential equation of the same form as (2) for  $\rho_a(r)$ . A general solution  $\rho_a^{(r)}(r)$  of this equation can be written as

$$\rho_a^{(r)}(r) = \frac{\rho_a^{(r)}(R)}{u(R, k)} \{u(r, k) + u_h(r, k)\} \quad \text{for } r < R, \quad (25)$$

where  $u(r, k)$  is the function in (3) and  $u_h(r, k)$  is any solution of the homogeneous equation  $Lu_h(r, k) = 0$ .

We have to make sure, however, that  $u_h(r, k)$  has to be chosen such that  $u_{nr}(r, k) = 0$ . To this end we combine (10) and (13):

$$u_{nr}(r, k) = \sum_n u_n(r) \int_0^R u(r', k) v_n(r') dr'. \quad (26)$$

Extracting  $u(r, k)$  from (25) and substituting into (26) we get

$$u_{nr}(r, k) = - \sum_n u_n(r) \int_0^R u_h(r', k) v_n(r') dr'. \quad (27)$$

$u_h(r, k)$  has to be chosen such that  $u_{nr}(r, k) = 0$ . From (27) we see using the biorthonormality of  $u_n, v_n$  that this can be true only if

$$\int_0^R u_h(r', k) v_n(r') dr' = 0 \quad \text{for every } n. \quad (28)$$

We will show that this can be true only if  $u_h(r, k) = 0$ . Using the contour integral technique of the previous section we find

$$\int_0^R u_h(r', k) \sin(ar') dr' = 0 \quad \text{for (almost) every } a, \quad (29)$$

and hence

$$u_h(r, k) = 0. \quad (30)$$

So (25) reduces to

$$\rho_a^{(r)}(r) = \frac{\rho_a^{(r)}(R)}{u(R, k)} u(r, k). \quad (31)$$

So the requirement that the physical source  $\rho_t(r)$  has to be purely radiating fixes the choice for  $\rho_a(r)$ . This has been achieved by projecting  $u(r, k)$  on a suitable space, i.e., the space spanned by the inhomogeneous modes belonging to  $\rho_a^{(r)}$ . As  $\rho_t(r)$  is a general source distribution, we have derived that any source distribution  $\rho_t(r)$  can be made purely radiating by a suitable choice for  $\rho_a(r)$ . In the next section we will present another approach inspired by the identification of  $u_{nr}(r, k)$  in Section 2. We will see that this approach leads to unexpected complications.

**4. Yet another identification of the radiative component of the field.**—In the preceding section we identified the nonradiative part of the field  $u(r, k)$  as the projection of

$u(r, k)$  on the space spanned by the nonradiating modes. In this section we take another, equally suggestive but less compelling, standpoint by defining the radiative part of  $u(r, k)$  as the projection of  $u(r, k)$  on the space spanned by the homogeneous modes  $v_n(r)$ . This is not so well justified as the approach of the previous section which is based on the rigorous boundary conditions for the nonradiating component. Now we get

$$u_{\text{rad}}(r, k) = \sum_n b_n(k) v_n(r) \quad \text{for } r < R. \quad (32)$$

Using the method of the previous section we find

$$\int_0^R u_{nr}(r, k) u_n(r) dr = 0 \quad (33)$$

by using (14) and  $b_n(k) = \int_0^R u(r, k) u_n(r) dr$ . By the method of the previous section we find

$$u_{nr}(r, k) = 0. \quad (34)$$

Therefore the field is purely radiating. We have, however, to make sure that  $u_{\text{rad}}$  satisfies

$$L u_{\text{rad}}(r, k) = \rho_t(r). \quad (35)$$

Substituting (32) we find, multiplying both sides of (35) by  $u_n(r)$  and integrating over  $r$  from 0 to  $R$  we get

$$\int_0^R \left[ u(r, k) - \frac{1}{k^2 - k_n^2} \rho_t(r) \right] u_n(r) dr = 0 \quad \text{for every } n. \quad (36)$$

So  $u_n(r)$  and consequently  $\rho_a(r)$  have to be chosen such that (36) is true. The reservation expressed in the beginning of this section becomes more visible now: the set of equations (36) has only one solution,  $\rho_a(r) = 0$  (a proof of this statement can be constructed using the technique at the end of Section 3). This is an absurd result, because we are unable to construct a biorthonormal set  $u_n, v_n$  from such a  $\rho_a$ :  $u_n(r)$  would be identically zero. Notice that  $u_{nr}(r, k)$  is identically zero, yet satisfies the boundary conditions (1). So the approach of this section leads to a dead end. We expect the approach of Section 3 to be the only consistent one.

**5. Discussion.**—While many examples of nonradiating sources have been mentioned in the literature (see, e.g., [4,5]), the decomposition of a wave field in a radiating and a nonradiating part ([3]) is much more complicated. We studied this problem using a biorthonormal system which we already introduced in [2]. This system has been shown to be exceptionally well tailored to the present problem. Using this tool we have shown that the decomposition of the wave field is nonunique as expressed by the presence of an arbitrary source distribution  $\rho_a(r)$  in the results of Section 3. Uniqueness of the reconstruction of the source distribution can be enforced only by using prior information (if available) or by imposing extra condition(s) like minimal-energy solution ([3]). Based upon the rigorous characterization of the nonradiating component of the wave field we imposed in Section 3 the condition that a purely radiating field does not couple to the nonradiating modes,

which implies that the expansion coefficients (15) are all zero. This condition then fully determines the arbitrary charge distribution:  $\rho_a^{(r)}(r) = \frac{\rho_a^{(r)}(R)}{u(R, k)} u(r, k)$  for  $r < R$ .

This conclusion is confirmed by the result of Section 4: the physical requirements of the problem lead to the conclusion that the approach of Section 4 leads to a dead end. Hence, although we expected quite some flexibility because of the arbitrary source  $\rho_a$ , the physical properties of the problem cannot be modified by formal manipulations: the physics leads to unique results.

We will now compare our approach with the one by Marengo and Ziolkowski [3]. They define a source distribution  $\rho(r)$  to lack a nonradiating part if it is orthogonal to any source distribution  $\rho_{nr}^a(r)$  in  $D$  which gives rise to a nonradiating field:

$$\int_0^R \rho(r) \rho_{nr}^a(r) dr = 0. \quad (37)$$

Translated in our notation,

$$\int_0^R \rho(r) L u_n(r) dr = 0 \quad \text{for every } u_n(r) \quad (38)$$

as  $L u_n(r)$  is the charge distribution for the nonradiating field  $u_n(r)$ . Equation (38) then leads to

$$\int_0^R L \rho(r) u_n(r) dr = 0, \quad (39)$$

which can be satisfied only when

$$L \rho(r) = 0. \quad (40)$$

The proof can be given by replacing  $u_{nr}$  in (33) by  $L \rho(r)$  and following the subsequent reasoning.

The purely radiating source in our approach is given by  $\rho_a^{(r)}$  in (25). However,  $L \rho_a^{(r)} = \frac{\rho_a^{(r)}(R)}{u(R, k)} \rho_t^{(r)}$  which is not identically zero. So we conclude that the approach in [3] differs from ours, the difference being due to different definitions: Ref. [3] starts from source distributions, and we start from the fields. Also, their sources differ from our fields by an operator  $L$ .

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